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AdS black holes and thermal Yang-Mills correlators

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Abstract: We study the real time correlators of scalar glueball operators for Yang-Mills theory at finite temperature in flat space. The analytic structure of the frequency space propagator in perturbative field theory is seen to be qualitatively different to the strong coupling results that may be obtained from perturbations about AdS black hole spacetimes: we find branch cuts rather than poles. This difference appears to persist away from the strict zero and infinite coupling limits, possibly suggesting a phase transition in large-N thermal $\mathcal{N} = 4$ SYM theory as a function of the 't Hooft coupling.

Keywords: Black Holes in String Theory, AdS-CFT Correspondence, Thermal Field Theory, 1/N Expansion.

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Contents

1. Introduction and summary

The AdS/CFT correspondence relates gravitational and field theories in complementary regimes of tractability [1]. This fact is both an important success and a significant limitation of the correspondence. On the one hand, such complementarity has allowed insight into the dynamics of strongly coupled gauge theories. However, if one wishes to study quantum gravity processes involving large curvatures in the interior but with asymptotically weak curvatures, such as evaporation of black holes, then the appropriate dual description is a strongly coupled field theory. Thus neither classical bulk physics nor perturbative boundary physics is able to provide significant insight into these processes.

It is therefore an important recent observation that gravitational physics in asymptotically AdS space with large curvatures everywhere appears to posses a similar phase structure to gravity with asymptotically weak curvatures [2 – 6]. In particular, there is a Hawking-Page phase transition [7] and a description in terms of an effective potential with two minima and a local maximum that may be interpreted as thermal AdS space, the large AdS black hole and the small AdS black hole, respectively. The emerging picture suggested that perturbative field theory could provide an inroad for studying quantum gravity pro-

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cesses that were at least qualitatively similar to those of real physical interest. Thus for instance one might hope that generalising the work of [3 − 5] to consider out of equilibirum real time thermal field theory on $S^3 \times \mathbb{R}$ could provide a framework for studying black hole evaporation.

The main thrust of this work is to exhibit in some detail one aspect at least in which black holes in strongly curved AdS space are qualitatively different to the more familiar weakly curved black holes. One consequence of these differences will be that the strongly curved black holes are not particularly black. Fields decay exponentially in weakly curved AdS black hole backgrounds [8, 9], with a timescale set by the black hole temperature. In free field theory on flat space, we see that the decay is generically only power law whilst including weak interactions results in a power law times a slow exponential decay over the plasmon damping timescale $1/(\lambda T \ln(\lambda^{-1}))$, where $\lambda \ll 1$ is the 't Hooft coupling. This behaviour is in line with recent observations in [10], who noted that black holes dual to weakly coupled field theories are not totally absorbent.

We will study the finite temperature, frequency space retarded propagators of the scalar glueball operators $\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ and $\mathrm{Tr}\, F_{\mu\nu}\widetilde{F}^{\mu\nu}$ at both strong and weak 't Hooft coupling in Yang-Mills theory in Minkowski spacetime. The $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory at finite temperature and in flat space is realised in a deconfined phase. In this context the operators we study are dual to the dilaton and axion (the RR scalar) fields, respectively, in the bulk AdS black hole background. At weak coupling, a gauge theory computation indicates that these propagators generically contain branch cuts. At strong coupling, the same propagators have poles and no branch cuts as found in, for example, [11, 13]. We build on previous computations of the strong coupling behaviour to argue that these poles persist under α' corrections without the introduction of branch cuts. This difference in analytic structure is directly related through a Fourier transform to the differing time decays for the fields mentioned in the previous paragraph. Interestingly, we find that the weak coupling frequency space propagators bear striking similarities to corresponding objects computed in topological black hole backgrounds such as the BTZ black hole.

The structure of the paper is as follows. In section 2 we compute precisely the free Yang-Mills glueball propagator on $\mathbb{R}^{1,3}$, giving both the time dependence and frequency space results. We also consider the effects of perturbative corrections to these results, noting in particular the resummations that are necessary due to hard thermal loops and collisions [14 − 16]. In section 3 we recall the prescription for computing finite temperature correlators from bulk black hole backgrounds [11, 17]. We use this prescription to compute and plot the retarded propagators in frequency space. In section 4 we prove that at least some of the strong coupling poles persist under α' corrections whilst branch cuts do not appear in their vicinity. Finally, section 5 contains a discussion in which we speculate on possible interpretations of our results, such as the existence of a phase transition in large-N thermal $\mathcal{N} = 4$ SYM theory at an intermediate 't Hooft coupling, and consequent future directions for research. Appendix A explains the effect on the correlators of including the thermal masses and widths, and appendix B contains computations of free thermal glueball correlators on $S^3 \times \mathbb{R}$.

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2. Thermal glueball propagators on $\mathbb{R}^{1,3}$

We begin by considering SU(N) $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory at finite temperature T in four dimensional Minkowski spacetime and at small 't Hooft coupling,

$$\lambda = g_{\rm YM}^2 N \ll 1\,. \tag{2.1}$$

In this section we will analyse the finite temperature, real time behaviour of two point correlators of the marginal, SO(6) singlet scalar glueball operators

$$\mathcal{G} = {\rm Tr}\, F_{\mu\nu}F^{\mu\nu}\,, \qquad \widetilde{\mathcal{G}} = {\rm Tr}\, F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \tag{2.2}$$

These are chiral operators in the $\mathcal{N} = 4$ theory. In the euclidean or static description wherein the thermal field theory is formulated on $\mathbb{R}^3 \times S^1$ these operators couple to scalar glueball states of the three dimensional effective theory.

The temporal decay of fluctuations or correlations in a *weakly coupled* Yang-Mills plasma at finite temperature will generally occur at several distinct time scales. At short time scales set by the inverse temperature, $t \sim 1/T$, the behaviour is completely determined by free field theory. Intermediate and late time behaviours are controlled by other physical scales such as the thermal gluon mass scale $(\sqrt{\lambda}T)^{-1}$, the gluon damping rate $(\lambda T)^{-1}$, etc. [15].

2.1 Short times or the free theory

For short time scales $t \sim T^{-1}$, and weak coupling, thermal correlators can be approximated by straightforward diagrammatic perturbation theory. Thus, at leading order we only need the quadratic terms in the Yang-Mills action and the glueball operator. Furthermore, we can focus attention solely on the transverse, physical degrees of freedom, since the longitudinal mode only mediates an instantaneous Coulomb interaction and does not affect the dynamics at any finite time $t > 0$. We also therefore don't need to worry about ghost fields. The time ordered two point function for the transverse gauge field is

$$\langle A_i^a(t,\mathbf{p})A_j^b(0,\mathbf{p})\rangle = D_{ij}^{>(ab)}(t)\;\Theta(t) + D_{ij}^{<(ab)}(t)\;\Theta(-t) \tag{2.3}$$

where $\mathbf{p}$ is the spatial momentum and [18, 19]

$$D_{ij}^{>(ab)}(t) = D_{ij}^{<(ab)}(-t) = i\left(\delta_{ij} - \frac{p_ip_j}{|p|^2}\right)\frac{\delta^{ab}}{2E}\left[(1+n(E))e^{-iEt} + n(E)e^{iEt}\right]. \tag{2.4}$$

Here $E = p \equiv |\mathbf{p}|$ and $n(E)$ is the Bose-Einstein distribution

$$n(E) = \frac{1}{e^{\beta E}-1}\,, \qquad \beta = \frac{1}{T}\,. \tag{2.5}$$

The glueball two point function with external momentum $\mathbf{p}$ is then straightforwardly worked out using Wick contractions to be

$$\begin{aligned}\langle \mathcal{G}^p(t)\mathcal{G}^p(0)\rangle &= \langle \widetilde{\mathcal{G}}^p(t)\widetilde{\mathcal{G}}^p(0)\rangle \equiv \int d^3x\; e^{-i\mathbf{p}\cdot\mathbf{x}}\langle \mathcal{G}(t,\mathbf{x})\mathcal{G}(0)\rangle \\ &= -\frac{N^2}{2\pi^3}\int\frac{d^3p'}{|\mathbf{p}'||\mathbf{p}-\mathbf{p}'|}\Big[\big(\mathbf{p}'^2(\mathbf{p}-\mathbf{p}')^2 + [\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}')]^2\big)F(|\mathbf{p}'|,t)F(|\mathbf{p}-\mathbf{p}'|,t) + \\ &\qquad + 2\,\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}')\dot{F}(|\mathbf{p}'|,t)\dot{F}(|\mathbf{p}-\mathbf{p}'|,t)\Big],\end{aligned} \tag{2.6}$$

where

$$F(E,t) = (n(E)+1)e^{-iEt} + n(E)e^{iEt}\,. \tag{2.7}$$

The $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ glueballs have the same free theory thermal two point functions. We will write $\mathcal{G}$ from now on for notational convenience.

We first consider the homogeneous mode with no external momentum, $\mathbf{p} = 0$. The relaxation of long-wavelength fluctuations or 'soft' modes with $p \ll T^{-1}$ can be non-trivially altered at late times by interactions, as we discuss below. Nonetheless in the limit of zero coupling, or for suitably short time scales in the interacting theory, the correlator at zero momentum, with $\mathbf{p} = 0$ in (2.6), becomes

$$\langle \mathcal{G}^0(\tau)\mathcal{G}^0(0)\rangle = -\frac{8N^2}{\pi^2}T^5 \int_0^\infty dx\, x^4 \frac{e^{2x}e^{-i2x\tau} + e^{i2x\tau}}{(e^x-1)^2}\,, \tag{2.8}$$

where we have introduced the dimensionless time $\tau = tT$ and the variable of integration $x = |\mathbf{p}'|/T$. The integral may be evaluated to give

$$\langle \mathcal{G}^0(\tau)\mathcal{G}^0(0)\rangle = \frac{N^2}{2\pi}T^5 \left[8\frac{d^3}{d\tau^3} + (2\tau + i)\frac{d^4}{d\tau^4}\right]\coth(2\pi\tau)\,. \tag{2.9}$$

As $\tau \to 0$ there is a $1/\tau^5$ divergence, which is the zero temperature power law behaviour. Importantly, this expression exhibits an exponential decay on time scales $\tau \sim 1$ or $t \sim T^{-1}$.

Using the above result we obtain the retarded propagator which is of particular physical interest and is given by

$$\langle \mathcal{G}^0(t)\mathcal{G}^0(0)\rangle_R = -i\Theta(t)\langle[\mathcal{G}^0(t),\mathcal{G}^0(0)]\rangle = \frac{N^2}{\pi}T\,\Theta(t)\frac{d^4}{dt^4}\coth(2\pi T t)\,. \tag{2.10}$$

The retarded propagators naturally occur in the framework of linear response theory wherein one probes the plasma with a weak disturbance and studies the response of the medium in a linearised approximation.

In the AdS/CFT correspondence, retarded thermal propagators of the boundary field theory are known to encode physical properties of AdS_5 black hole backgrounds, at least in the classical gravity approximation which is dual to strongly coupled field theory. Specifically, poles of retarded propagators in the frequency domain correspond to quasinormal frequencies of AdS black holes [20, 11]. The scalar glueball operators $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are dual to the dilaton and axion in the bulk, respectively. The associated retarded correlators *at strong coupling* encode quasinormal frequencies of massless minimally coupled scalar perturbations about the AdS_5 black hole background. We will return to the gravitational computation of correlators in later sections of this paper.

A characteristic feature of scalar perturbations about AdS black hole spacetimes is that they always decay exponentially [8, 9]. The exponential decay (2.10) that we have encountered in the free theory might seem to be consistent with this feature, although it is in a regime where the dual geometry is very strongly curved. However, the exponential decay of quasinormal perturbations in AdS black hole spacetimes represents the decay of perturbations in the boundary thermal theory and its consequent approach towards equilibrium. The exponential decay that we see for the homogeneous mode in the free

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theory is not, strictly speaking, an approach to equilibrium since scattering and collisional processes are absent in this limit. The decaying transient that we see here arises from interference effects between the different modes in (2.8). In fact we will see shortly that all the inhomogeneous modes of (2.6) in the free theory have oscillating power law decays instead, a fact which will have a simple physical explanantion.

The qualitative similarities and dissimilarities between the field theory at weak and strong 't Hooft couplings are naturally captured by the analytic structure of the retarded propagator in frequency space. Fourier transforming equation (2.10) we find

$$G^0_R(\omega) = -\frac{N^2}{2\pi^2}\,\omega^4\left[\psi\left(\frac{-i\,\omega}{4\pi T}\right) - \frac{1}{3}\left(\frac{2\pi T}{\omega}\right)^2 - \frac{2}{15}\left(\frac{2\pi T}{\omega}\right)^4 + i\frac{2\pi T}{\omega}\right], \qquad (2.11)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$, the logarithmic derivative of the Gamma function. In the zero temperature limit, using Stirling's approximation for $\Gamma(z)$ at large z, we have

$$G^0_R(\omega) \approx -\frac{N^2}{4\pi^2}\,\omega^4(\log\frac{\omega^2}{\mu^2} - i\pi\,\mathrm{sgn}\,\omega) + \cdots \quad \text{as} \quad \frac{\omega}{T} \to \infty\,, \qquad (2.12)$$

which is the known zero temperature result (see e.g. [11]).

The Fourier transformed homogeneous correlator (2.11) is analytic in the upper half plane $\mathfrak{Im}\,\omega > 0$. This is always true for retarded propagators with our conventions. On the other hand the function has an infinite set of simple poles along the lower imaginary axis at

$$\omega_n = -4\pi i T n\,, \qquad n = 1, 2\ldots \qquad (2.13)$$

These poles, illustrated in figure 1, lead to the exponential decay in time of (2.10). This analytic structure at zero 't Hooft coupling will of course be modified by a small nonzero value of λ and we will discuss the modifications in more detail below. Perhaps unsurprisingly, the analytic structure at $\lambda = 0$ bears little resemblance to that at $\lambda \to \infty$ wherein the retarded correlator has an infinite number of simple poles at locations that at large n tend to $\omega_0^\pm \pm 2\pi T\, n\,(1 \mp i)$ where $\omega_0^\pm \approx \pm 1.2139 - 0.7775i$ [11, 12]. The strong coupling poles are sketched in figure 2. The strong-weak coupling discrepancy will be substantially more severe for the inhomogeneous modes.

It is interesting to note that the frequency space propagator (2.11) is, up to a dimension dependent phase space factor, precisely the function encountered for the propagator at zero spatial momentum in the general BTZ black hole background dual to a two dimensional CFT [11]. Drawing further on this analogy, perhaps this suggests that like the BTZ black hole, the strongly curved five dimensional background is a kind of topological black hole (c.f. higher dimensional AdS space with global identifications in for instance [21 – 23]). From the free field theory perspective, the similarity to a two dimensional CFT and the BTZ black hole arises simply because free field correlators are completely determined by the scaling dimensions of the operators in question and the effect of finite temperature is to require an infinite sum over images that renders the correlators periodic in imaginary time.

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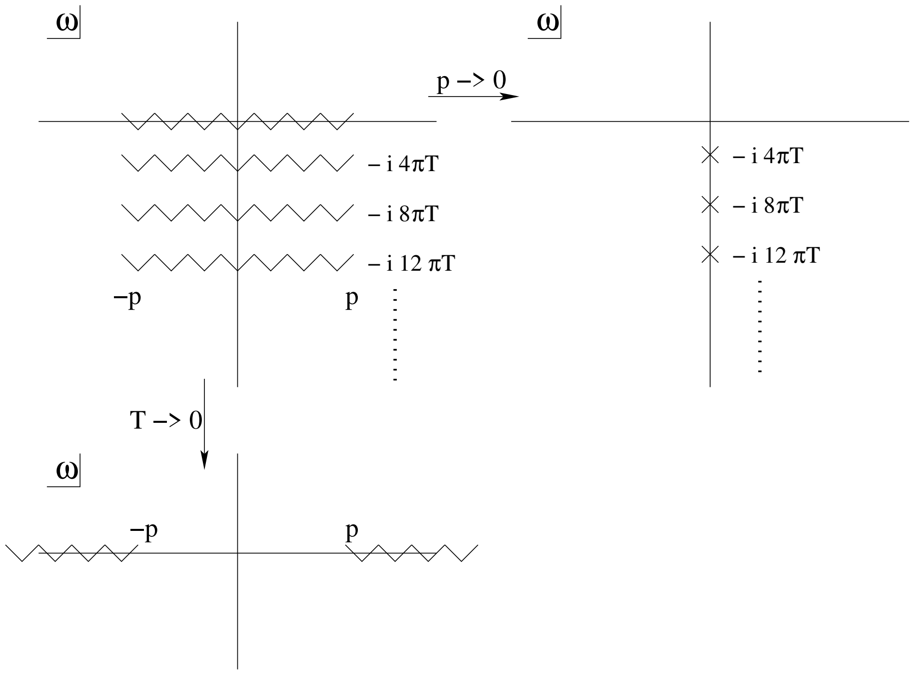

Figure 1: Poles and branch cuts of the free theory propagator $G^p_R(\omega)$. Top left is the full result with momentum p. Top right is the homogeneous mode with $p = 0$. Bottom left is the zero temperature limit with momentum p.

The inhomogeneous contribution to the propagator (2.6), with spatial momentum $\mathbf{p}$, may also be calculated. Introducing the dimensionless momentum $\mathbf{k} = \mathbf{p}/T$, the retarded propagator is given in terms of the integral

$$\langle \mathcal{G}^p(\tau)\mathcal{G}^p(0)\rangle_R = \frac{-N^2\Theta(\tau)}{4\pi^2}\frac{T^5}{k}\int_k^\infty du \int_{-k}^k dv \Bigg[\frac{(u^2-k^2)^2(1-e^u)}{(e^{(u+v)/2}-1)(e^{(u-v)/2}-1)}\sin(u\tau) - \frac{(k^2-v^2)^2 e^{u/2}(e^{v/2}-e^{-v/2})}{(e^{(u+v)/2}-1)(e^{(u-v)/2}-1)}\sin(v\tau)\Bigg]. \quad (2.14)$$

The double integral (2.14) may be evaluated, partly using contour integration, to give

$$\langle \mathcal{G}^p(t)\mathcal{G}^p(0)\rangle_R = \frac{N^2\Theta(t)}{\pi^2}\frac{T}{p}\left(p^2 + \frac{d^2}{dt^2}\right)^2 \left[\frac{\sin pt}{t}\pi\coth(2\pi t T) - \frac{\sin pt - pt\cos pt}{2t^2 T}\right]. \quad (2.15)$$

As $t \to \infty$ this propagator decays as an oscillatory power law rather than exponentially. The physical reason for this is that these modes carry a conserved momentum which in the free theory cannot dissipate into other modes. In the strongly coupled theory ($\lambda \to \infty$), dual to the classical black hole, the momentum can quickly dissipate into other modes and so one expects an exponential decay in this case, as indeed occurs. A small but nonzero λ will also modify this power law decay at intermediate and late times as we discuss below.

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The propagator (2.15) may be Fourier transformed to frequency space to yield

$$G_R^p(\omega) = -\frac{N^2}{\pi^2}(p^2-\omega^2)^2\left[\frac{1}{2}+\left(i\frac{\pi T}{2p}-\frac{\omega}{4p}\right)\log\frac{\omega+p}{\omega-p}+i\frac{\pi T}{p}\log\frac{\Gamma\left(\frac{-i(\omega+p)}{4\pi T}\right)}{\Gamma\left(\frac{-i(\omega-p)}{4\pi T}\right)}\right]+ \\ +\frac{N^2}{\pi^2}\left[\frac{2\pi^2T^2}{3}(\omega^2-p^2)+\frac{16\pi^4T^4}{15}+\frac{p^2}{6}\left(\frac{7p^2}{5}-\omega^2\right)\right]. \tag{2.16}$$

It is easily seen that (2.16) reduces to (2.11) in the vanishing momentum $p \to 0$ limit. In the zero temperature limit we also recover the wellknown Lorentz invariant result, providing a quick check of (2.16)

$$G_R^p(\omega) = \frac{-N^2}{4\pi^2}(p^2-\omega^2)^2\log\frac{p^2-\omega^2}{\mu^2}. \tag{2.17}$$

The central point to note here is that the frequency space result (2.16) has only branch cuts and no poles, as illustrated in figure 1. The infinite set of simple poles of the zero momentum correlator (2.11) have smeared out into branch cuts stretching between $\omega = -4\pi iTn - p$ and $\omega = -4\pi iTn + p$ with $n = 1, 2, \ldots$. In the time domain, these branch cuts centred around points on the negative imaginary axis translate into exponential falloff accompanied by power law and oscillatory corrections. In addition to these, there is also a branch cut on the real axis between $\omega = -p$ and $\omega = +p$ and it is this branch cut which is responsible for the leading oscillatory power law behaviour in time in (2.15).

These results may be contrasted with the analytic structure emerging from dual gravitational computations that describe the strongly coupled regime, e.g. [13, 11]. The strong coupling results are sketched in figure 2, displaying an infinite set of simple poles corresponding to the quasinormal frequencies of the AdS_5 black hole background. These asymptote to $\omega_0^{\pm}(p) \pm 2\pi Tn(1 \mp i)$. Note that both the real and imaginary parts of the quasinormal frequencies grow linearly with $n = 1, 2, \ldots$.

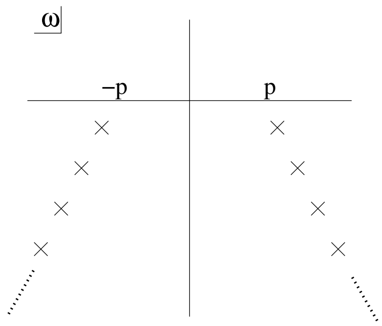

Figure 2: Poles in the strongly coupled propagator [13].

Thus we can see that correlators in the $\lambda = 0$ theory exhibit features that are qualitatively different to those of the $\lambda \to \infty$ theory. As we will argue in the following section, moving slightly away from the $\lambda = 0$ point does not change this qualitative picture drastically, although it does affect the late time behaviour of the correlator (2.15) causing it to decay exponentially. While branch cuts in frequency space induce power laws in the time domain, the endpoints of the branch cuts dictate the exponential decay constants and/or oscillatory behaviour. For fixed p the real parts of the branch point locations of (2.16) are fixed while the imaginary part increases with n. Following our previous comments concerning parallels with the BTZ black hole, it is tantalising to note that quasinormal frequencies of topological AdS black holes appear to have a similar property, namely that for fixed spatial momentum their real parts are constant while the imaginary parts take an infinite set of discrete values [24].

Finally, we have already noted that the operator $\widetilde{\mathcal{G}}(\mathbf{x},t)$ in (2.2) has exactly the same two point function as $\mathcal{G}$ in free field theory. This operator has the quantum numbers of the 0^{-+} scalar glueball, and at strong coupling it is dual to the axion (the RR scalar) in supergravity. We will see shortly that this implies that also at strong 't Hooft coupling the operators $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ share the same correlators. At finite values of λ however, we expect them to differ quantitatively.

2.2 Late times and weak coupling $\lambda \neq 0$

We now turn to the effects of nonzero 't Hooft coupling λ on the late time behaviour of the correlators above. In the interacting $\mathcal{N}=4$ SYM theory, the correlators of interest receive higher order contributions from gluon self-interactions and from scalar and fermion matter fields in the adjoint representation of the gauge group. Since the thermal bath breaks SUSY and conformal invariance, in what follows it is sufficient to focus on the known general qualitative aspects of time dependent correlators in a weakly interacting Yang-Mills plasma in the deconfined phase. In the weakly coupled plasma there is a hierarchy of energy scales, $(T, \sqrt{\lambda}T, \lambda T, \ldots)$ corresponding to distinct physical phenomena that dictate the temporal behaviour of correlators over the associated characteristic time scales.

In order to go beyond the leading perturbative result it is necessary to implement the finite temperature improvement of resumming the hard thermal loops [14]. Essentially this means that for the internal lines with momenta $\mathbf{p}'$ or $\mathbf{p}-\mathbf{p}'$ in the above one loop calculation, we should include the thermal self-energy corrections. Specifically, for the hard momenta of order T which dominate the graph, this self-energy correction leads to an effective thermal mass $m_T \sim \sqrt{\lambda}T$. In addition there is also a higher order effect leading to a thermal width, or more precisely the plasmon damping rate for hard gluons, $\Gamma \sim \lambda T \ln(\lambda^{-1})$. This resummation has a significant effect on homogeneous correlators or those with 'soft' external momentum $p \lesssim m_T \sim \sqrt{\lambda}T$. For 'hard' modes, with external momentum $p \gtrsim T$, the effect of the hard thermal loop resummation remains perturbatively small and does not significantly alter the early time oscillatory power law behaviour found in (2.15).

First consider the effect of incorporating thermal masses for internal lines. This will shift the threshold branch points at $\omega = \pm p$ to $\omega = \pm\sqrt{p^2 + 4m_T^2}$. The thermal mass thus changes the analytic structure of the homogeneous correlator with $p = 0$ (2.11) by introducing a branch cut between $\omega = +2m_T$ and $\omega = -2m_T$ which leads to oscillatory power law decays in time. This change in the analytic structure can be inferred from (2.6) by setting $p = 0$ and using the Bose-Einstein distribution for a massive particle. We perform this computation in appendix A.

It is easy to see, following the example in appendix B of [15], that this inclusion of the thermal mass leads to a power law decay with oscillatory behaviour for time scales $t > (\sqrt{\lambda}T)^{-1}$, where the oscillations are controlled by the thermal mass m_T. Hence the short time exponential decay of the homogeneous correlator (2.10) turns into an oscillatory power law on time scales of order $(\sqrt{\lambda}T)^{-1}$. This power law behavior kicks in when the free field correlator has decayed to $\sqrt{\lambda}$ times its value at $t = 0$.

However, including only thermal masses is, strictly speaking, a collisionless approximation from the point of view of quasiparticles in the plasma although it includes quantum field theory interactions. What happens eventually to all correlators at late times is actually determined by scattering and collisions of quasiparticle states. A detailed quantitative treatment of this question is beyond the scope of this work. However, much can be learned by following the arguments in [15]. Incorporating the thermal widths of intermediate states, the imaginary part Γ of the self energy, makes internal propagators decay on time scales of order $(\lambda T \ln(\lambda^{-1}))^{-1}$ and hence the correlators themselves decay exponentially on these time scales. This is a somewhat subtle point as discussed in [15]. The effect of incorporating the width, or plasmon damping rate, can in certain cases actually cancel out between different diagrams in resummed perturbation theory so that the exponential decay time scale is longer and is dictated by different physics. In particular, this occurs for correlators involving conserved currents wherein the time scale for exponential decay is controlled by that of large angle scattering $\sim (\lambda^2 T \ln(\lambda^{-1}))^{-1}$ and can be seen using an effective kinetic theory description [16].[1] Since the scalar glueball operators are not associated to any conserved currents or charges, we expect them to decay on time scales set by plasmon damping which is also referred to in [15] as the time scale for 'any-angle scattering'.

The picture that thus emerges when finite coupling λ is considered is that the analytic structure of the glueball two point function in frequency space is qualitatively similar to figure 1, with an extra branch cut about the origin along the real axis in the homogeneous case. But this description cannot account for the late time exponential decay argued for above since the branch cut along the real axis in the frequency plane seen in the lowest order approximation would lead to oscillatory power laws at late times. This discrepancy suggests the following possibility:

(1) When all relevant perturbative corrections are accounted for, the branch points on the real frequency axis are shifted downwards into the lower half plane by an amount $\sim i\lambda T \ln(\lambda^{-1})$. A naïve calculation illustrating this effect is given in appendix A.

This analytic structure for frequency space propagators would be consistent with power law tails at intermediate times accompanied by exponential decay times power laws at late times. A confimation of this possibility would require a detailed consideration of the correlators with full self-energy resummations for the internal lines.

However, finite temperature perturbation theory suffers from IR divergences at higher loops. These are usually cut off by hand at the nonperturbative magnetic scale λT. Such a regularisation is the source of the $\log \lambda^{-1}$ enhancement of the plasmon damping rate, for instance. The presence of nonpertubrative physics means that we cannot rule out the following further possibility:

(2) The branch cut seen in the lowest order approximation from $\omega = \pm 2m_T$ for the homogeneous correlator, and from $\omega \approx \pm p$ for the inhomogeneous cases, might split

[1]Correlators for conserved currents in a finite N theory eventually decay via hydrodynamic power law tails which are not visible in the large-N limit [25]. These $1/N$ suppressed power law tails correspond to non-analyticities in the frequency space correlators at small ω.

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into a series of poles with nonperturbatively small separations, which then move off the real axis upon inclusion of all perturbative and nonperturbative corrections in the large-N limit of the $\mathcal{N} = 4$ theory.

Whilst we have no evidence in favour of this second possibility, it would reproduce the expected intermediate and late time behaviour in the weakly coupled plasma, and would also allow a smooth interpolation to the strongly coupled gravity dual picture discussed in the following section.

Current perturbative results for real time response functions in weakly coupled plasmas in the deconfined phase generically contain branch cuts of the kind we have been discussing and further give rise to branch cuts in correlators from zero frequency (see e.g. [26, 27]) and no poles.

Our discussion in this section has consequences for bulk physics. The strong coupling exponential decay on timescales $t \sim T^{-1}$ is naturally interpreted gravitationally as being due to matter falling into the black hole. The much slower decays we have just described at weak coupling therefore suggest that black holes in highly curved AdS backgrounds are less absorbent. This phenomenon is consistent with the recent comments in [10].

3. Strong coupling correlators from gravity

A prescription for computing thermal two point correlators in strongly coupled $\mathcal{N} = 4$ SYM theory was developed in [11, 17]. This method has allowed the computation of poles in the Minkowski space retarded thermal propagator via bulk quasinormal modes [13, 28], with particular success in the hydrodynamic limit [29].

In this section we review the bulk prescription and use it to produce plots of the thermal propagator as a function of frequency ω. These weak curvature results describe the strongly coupled SYM theory. We will compare with plots for the free theory correlators following from our results in the previous section, highlighting various qualitative differences. In the following section we will show that the strong coupling correlators do not have branch cuts, as was implicitly anticipated in [13, 28] and thus confirm figure 2. Our results on the absence of branch cuts will extend to include α' corrections.

For notational clarity when discussing gravitational computations, we shall absorb T into the frequency: $\omega/T \to \omega$.

As we wish to compare with field theory on $\mathbb{R}^{1,3}$, the bulk is given by the near horizon geometry of a stack of non-extremal D3 branes

$$ds^2 = \frac{R^2}{z^2}\left[-f(z)dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)}\right] + R^2 d\Omega_5^2\,, \tag{3.1}$$

where as usual $d\Omega_5^2$ is the round metric on a five sphere, R sets the AdS scale and

$$f(z) = 1 - \frac{z^4}{z_H^4}\,, \tag{3.2}$$

with the horizon $z_H = \beta/\pi$.

The prescription may now be stated as [11, 17]

$$G_R(\omega) = K\sqrt{-g}g^{zz}\phi_\omega(z)\partial_z\phi^*_\omega(z)\big|_{z_B\to 0}\,, \tag{3.3}$$

where K is a normalisation constant. The normalisation is known from the bulk action [11] or alternatively can be fixed by the normalisation of the zero temperature behaviour $\omega^4\log\omega$ as $\omega\to\infty$. The zero temperature result will agree with the boundary free field theory computation because the $\mathrm{Tr}\,F^2$ operator is BPS. The bulk modes $\phi(z)$ are evaluated near the boundary $z_B \ll R$ where they are required to satisfy $\phi(z_B)=1$. They further satisfy incoming wave boundary conditions at the horizon

$$\phi_\omega(z) \sim (z_H - z)^{-i\omega/4\pi} \quad \text{as} \quad z\to z_H\,. \tag{3.4}$$

As usual in computing boundary correlators using the AdS/CFT correspodnence, the expression (3.3) will contain volume divergences that correspond to UV divergences in the dual theory. In order to meaningfully compare bulk and boundary results, we need to fix a regularisation scheme. We do this as follows: Firstly in (3.3) we minimally subtract off all the divergent terms as $z_B\to 0$. This gives a finite propagator whose only scheme dependence is due to the ambiguous subtration of a logarithmic divergence $\sim\omega^4\log z_B/\mu$. In order to extract scheme independent information we then subtract off from the propagator the terms that diverge like $\omega^4\ln\omega$ and ω^4 as $\omega\to\infty$. This subtraction corresponds to removing the zero temperature result. We will perform the same subtractions on our propagators from the previous section.

The bulk field dual to the glueball operator $\mathrm{Tr}\,F^2$ is the dilaton, whilst the field dual to $\mathrm{Tr}\,F\widetilde{F}$ is the axion. Both of these fields are minimally coupled scalars and therefore have a mode equation in the black brane background (3.1) given by [11, 13]

$$\phi'' - \frac{1+u^2}{u(1-u^2)}\phi' + \frac{w^2/4\pi^2}{u(1-u^2)^2}\phi - \frac{p^2/4\pi^2}{u(1-u^2)}\phi = 0\,, \tag{3.5}$$

where the coordinate $u = z^2/z_H^2$. By solving this equation numerically and varying ω and q, we can plot the propagator (3.3). The real part of the propagator has a more interesting structure than the imaginary part, so we plot this. Furthermore, the real part of $G_R(\omega)$ is symmetric on the real axis, so we restrict plots to $\omega\geq 0$. It is curious that these two operators have exactly the same thermal propagator both at zero and at strong coupling.

Figures 3 and 4 show the propagator for the homogeneous mode as computed in free field theory (2.11) and from the bulk using (3.3), respectively. As discussed above, the zero temperature result has been subtracted to ensure scheme independence

$$\Delta G^p_R(\omega) = G^p_R(\omega) - \#(p^2-\omega^2)^2\log\left(p^2-\omega^2\right)\,. \tag{3.6}$$

Although the plots show some similarity, and curiously both lack an ω^2 growth as $\omega\to\infty$, they remain fairly distinct, consistent with their differing analytic structure off the real axis.

Figures 5 and 6 show the effect of including a momentum p in the propagator in free field theory and in the bulk, respetively. In this case we have plotted for both positive and negative ω to emphasise the features of the plots.

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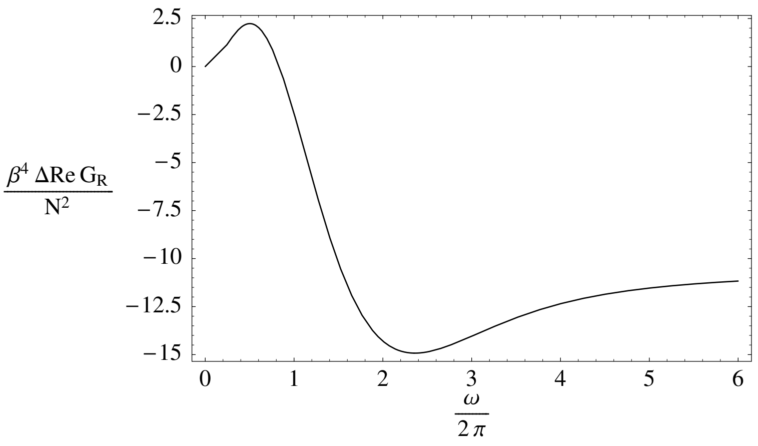

Figure 3: The real part of $\Delta G^0_R(\omega)$ as computed in free field theory.

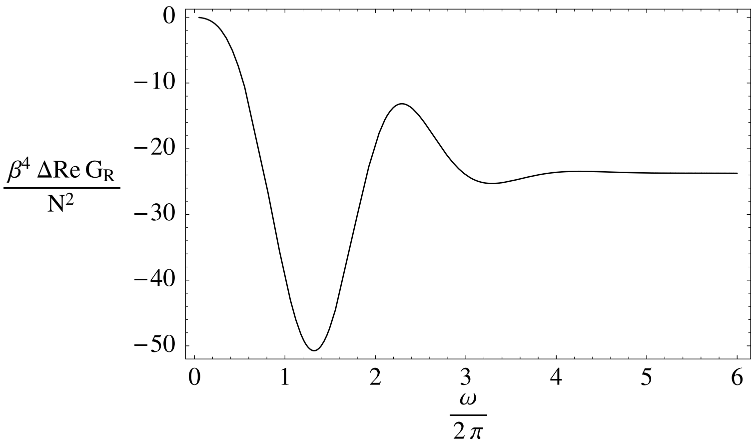

Figure 4: The real part of $\Delta G^0_R(\omega)$ as computed from the bulk supergravity.

The main features in the free theory plot of figure 5 are a positive p^4 growth with momentum along the $\omega = 0$ axis which is apparent in (2.16) and a negative ω^2 growth which becomes more significant at larger p. The transition between these behaviours occurs around $\omega \sim \pm p$ where the branch points are located.

In the bulk computation shown in figure 6 we see that the peak and trough present already in the homogeneous case, figure 4, become more pronounced once p is included and move linearly like $\omega \sim \pm p$ as p increases. This is consistent with these peaks being resonances of the nearest quasinormal pole to the real ω-axis, see figure 2. It was shown in [13] that as p is increased the lowest poles move closer to the real axis and tend towards $\omega = \pm p$.

One can numerically check that there is no branch point at, for instance, $\omega = \pm p$ in the bulk results by computing the propagator along the circles

$$\omega = \pm p + e^{i\theta}\,, \qquad \theta \in [0, 2\pi]\,, \tag{3.7}$$

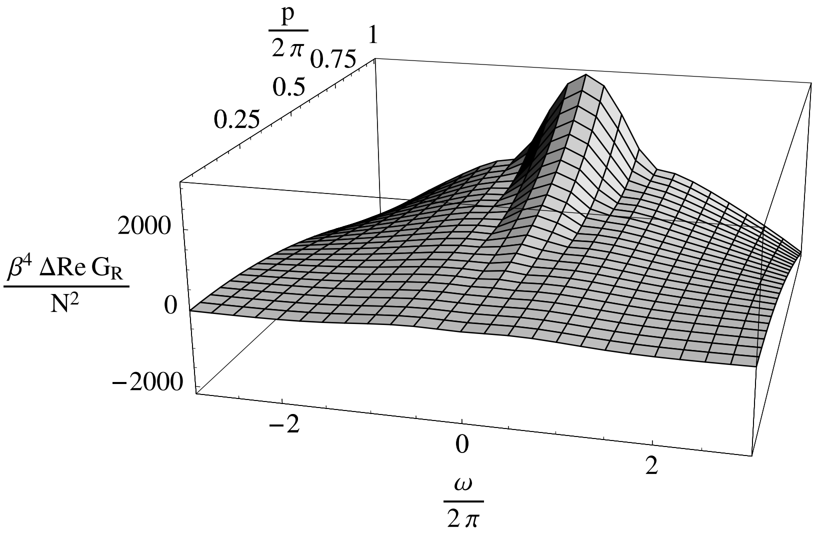

Figure 5: The real part of $\Delta G_R^p(\omega)$ as computed in free field theory.

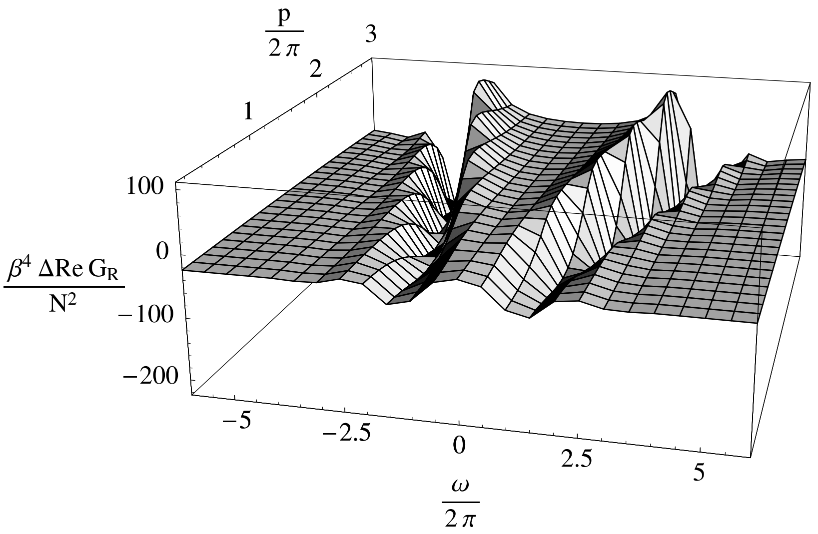

Figure 6: The real part of $\Delta G_R^p(\omega)$ as computed from the bulk supergravity.

and verifying that both real and imaginary parts of the propagator are continuous. In the following section we will prove the absence of branch points in some regions of the complex ω plane. This argument will then extend to include α' corrections.

4. Persistence of poles under α' corrections

To move away from the strict strong coupling limit, we need to consider string theory α' corrections to the gravity background and to the equations of motion for the fluctuations. We will be able to show rigorously that at least some of the poles in the retarded glueball propagator that are found in the strong coupling limit persist under α' corrections without becoming branch points. The argument does not rely on the precise form of the α' corrections, which is fortunate as even the leading order corrections to the type-IIB string theory action are not known completely and hence precise calculations using such corrections are not possible in our context, despite some assumptions to the contrary in the literature.

The computation of the retarded glueball propagator from the bulk follows the same logic as previously. One takes the corrected background and considers linearised fluctuations of the dilaton or axion about this background. The fluctuation equations will be higher order and will therefore have more than two independent solutions. However, only two of these solutions will deform into the uncorrected solutions in the $\alpha' \to 0$ limit. These are the modes dual to the glueball operator. The other modes are stringy degrees of freedom arising in the α' corrected action, these are dual to other boundary operators.

The boundary correlator is then calculated as usual by considering the on shell corrected bulk action as a function of the boundary field values. As we tend towards the boundary, $u \to 0$, the leading u dependence of the field must be the same as in the uncorrected case, otherwise the α' corrections would not be subleading and the perturbation expansion would not be reliable. Thus we have from (3.5)

$$\phi_\omega(u) = A(\omega)\,(1 + \cdots) + B(\omega)\left(u^2 + \cdots\right) \qquad \text{as} \quad u \to 0\,. \tag{4.1}$$

The coefficients $A(\omega)$ and $B(\omega)$ are fixed by the incoming wave boundary condition at the horizon (3.4). The fact that the u dependence near the boundary is as in the uncorrected case, although the ω dependence of (4.1) is of course different, implies that the $G_R(\omega)$ correlator has the same structure as previously in (3.3)

$$G_R(\omega) \sim \left.\frac{\phi_\omega \partial_u \phi^*_\omega}{\phi_\omega \phi^*_\omega}\right|_{u=u_B\to 0} \sim \frac{B(\omega)}{A(\omega)}\,. \tag{4.2}$$

This expression should be multiplied by functions with u dependence, and the evaluation as $u \to 0$ will require regularisation. By dividing by $\phi_\omega\phi^*_\omega$ we have explicitly imposed the additional normalisation $\phi(u_B) = 1$ at the boundary cutoff. The expectation that the result reduces to a $\phi_\omega\partial_u\phi^*_\omega$ term is realised explicitly in [30] who consider a partial set of α' corrections to the Yang-Mills shear viscosity.

The expression (4.2) was a basic connection between bulk quasinormal modes and poles in the dual propagators observed in [11]. A quasinormal mode in asymptotically AdS space satisfies incoming boundary conditions at the horizon and is normalisable as $u \to 0$. This will occur at frequencies where $A(\omega) = 0$, which hence correspond to poles in $G_R(\omega)$. The corrected fluctuation equation will have quasinormal modes and hence dual poles that are simply deformations of the strict strong coupling poles. However, the nontrivial question which we will now address is whether these poles can furthermore become branch points and whether extra branch points can appear. We are able to give a definite negative answer in some cases, and expect this result to hold in general.

Branch cuts in the ω plane may appear in $A(\omega)$ and $B(\omega)$ as defined in (4.2). The analytic properties of these functions are inhereted from the analytic properties of $\phi_\omega(u)$ for small u as a function of ω. Recall that ω specified the boundary conditions as $u \to 1$. We shall shortly adapt some arguments from [31] to prove some analyticity of $\phi_\omega(u)$ in ω

$$-2\pi < \mathfrak{Im}\,\omega \quad \Rightarrow \quad \text{analytic}\,. \tag{4.3}$$

Thus we gain a strip in the lower half plane in which $\phi_\omega(u)$ is analytic and hence $G_R(\omega)$ does not have branch points. However, it was found in [13] that as p becomes large, the imaginary part of the location of the lowest quasinormal pole moves towards the real axis. In particular, there exist infinitely many values of p for which the lowest quasinormal pole is within the range (4.3). The location of these poles will only be altered by amounts of order α' and so will remain in this strip. However, there can be no branch points in this region and therefore these poles cannot have become branch points and no new branch points can have appeared in this region. This is the desired result. There does not seem to be a reason why these larger p poles should be special at strong coupling, so we expect that the absence of branch cuts holds in general. In any case, for the logic of this paper it is sufficient for us to prove the persistence of some poles.

We first prove analyticity of $\phi_\omega(u)$ in the region (4.3) for the uncorrected potential. It is convenient to work in Regge-Wheeler coordinates, so that equation (3.5) assumes a Schrödinger form. Set

$$dr_* = \frac{1}{2\pi}\frac{du}{u^{1/2}(1-u^2)}\,, \tag{4.4}$$

which implies the range $0 \leq r_* < \infty$. Now let

$$\psi = u^{-3/4}\phi\,. \tag{4.5}$$

The bulk equation (3.5) becomes

$$-\frac{d^2\psi}{dr_*^2} + V(r_*(u))\psi = \omega^2\psi\,, \tag{4.6}$$

where

$$V(r_*(u)) = p^2(1-u^2) + \frac{\pi^2}{u}\left[4 - \frac{(3u^2+1)^2}{4}\right]. \tag{4.7}$$

The only property we need of this potential is the exponential decay towards the horizon

$$V(r_*) \sim e^{-4\pi r_*} \quad \text{as} \quad r_* \to \infty\,. \tag{4.8}$$

The incoming boundary condition at the horizon (3.4) becomes

$$\psi_\omega \sim e^{i\omega r_*} \quad \text{as} \quad r_* \to \infty\,. \tag{4.9}$$

Now rewrite the Schrödinger equation (4.6) as an integral equation incorporating the initial condition (4.9)

$$\psi_\omega(r_*) = e^{i\omega r_*} - \frac{1}{\omega}\int_{r_*}^{\infty} dx \sin\omega(r_* - x)V(x)\psi_\omega(x)\,. \tag{4.10}$$

The integral equation may be solved using a Born series

$$\psi_\omega = \sum_{n=0}^{\infty}\psi_\omega^{(n)}\,, \tag{4.11}$$

where

$$\psi_\omega^{(0)}(r_*) = e^{i\omega r_*}\,,$$
$$\psi_\omega^{(n)}(r_*) = -\frac{1}{\omega}\int_{r_*}^{\infty} dx \sin\omega(r_*-x)V(x)\psi_\omega^{(n-1)}(x)\,. \tag{4.12}$$

Using the inequality

$$|\sin\omega(r_*-x)| \leq e^{-|\,\mathfrak{Im}\,\omega|(r_*-x)} \quad \text{for} \quad x > r_*\,, \tag{4.13}$$

it is straightforwardly shown that

$$\sum_{n=0}^{\infty}\left|\psi_\omega^{(n)}(r_*)\right| \leq e^{-r_*\,\mathfrak{Im}\,\omega} - e^{-r_*|\,\mathfrak{Im}\,\omega|} + \\ +e^{-r_*|\,\mathfrak{Im}\,\omega|}\exp\left[\frac{1}{|\omega|}\int_{r_*}^{\infty} dx|V(x)|e^{x(|\,\mathfrak{Im}\,\omega|-\mathfrak{Im}\,\omega)}\right]. \tag{4.14}$$

Therefore the Born series is absolutely convergent if

$$\int_{r_*}^{\infty} dx|V(x)|e^{x(|\,\mathfrak{Im}\,\omega|-\mathfrak{Im}\,\omega)} < \infty\,. \tag{4.15}$$

Given the exponential decay of the potential (4.8), it follows that the integral (4.15) is finite in the region (4.3). The consequent absolute convergence of the Born series establishes continuity of ψ_ω in this region. Analyticity follows from performing an entirely analogous computation for the derivative of ψ_ω with respect to ω. Thus we prove the analyticity claim above for the uncorrected case. Whilst the lower limit of integration, r_*, is small, we may keep it finite and therefore do not need to worry about divergences as $r_* \to 0$.

To extend this result to include α' corrections we need to perturb the solution and the equation. In terms of the solution we have

$$\psi \to \psi + \epsilon\psi'\,, \tag{4.16}$$

where ϵ is a small parameter controlling the α' corrections. The zeroth order solution ψ satisfies (4.6) whilst the perturbation satisfies

$$-\frac{d^2\psi'}{dr_*^2} + V(r_*(u))\psi' + \mathcal{O}\psi = \omega^2\psi'\,, \tag{4.17}$$

where $\mathcal{O}$ is a higher order differential operator which contains the effects of the α' corrections. This equation is the same as that satisfied by the zeroth order solution but with an extra inhomogeneous term. We do not need to know the precise form of $\mathcal{O}$, but note that consistency of the α' expansion requires

$$|V\psi'|, |\mathcal{O}\psi| \ll \frac{1}{\epsilon}|V\psi| \qquad \text{as} \quad r_* \to \infty\,. \tag{4.18}$$

The integral equation satisfied by the perturbation is

$$\psi'_\omega(r_*) = e^{i\omega r_*} - \frac{1}{\omega}\int_{r_*}^{\infty} dx \sin\omega(r_*-x)\left[V(x)\psi'_\omega(x) + \mathcal{O}\psi_\omega(x)\right]. \tag{4.19}$$

We may again solve this equation using a Born series. All the terms are as previously in (4.12) except for the first term which is now

$$\psi'^{(0)}_{\omega}(r_*) = e^{i\omega r_*} - \frac{1}{\omega}\int_{r_*}^{\infty} dx \sin\omega(r_* - x)\mathcal{O}\psi_\omega(x)\,. \tag{4.20}$$

The zeroth order solution, $\psi_\omega(x)$, satisfies the inequality

$$|\psi_\omega(x)| \le k e^{-x\,\mathfrak{Im}\,\omega}\,, \tag{4.21}$$

where k is some constant. The bound (4.21) follows from the asymptotic behaviour of $\psi_\omega(x)$ (4.9) and the absence of divergences at finite values of x and as $x \to 0$. Using this bound together with the inequalities (4.18) it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\psi'^{(n)}_{\omega}(r_*)\right| &\le e^{-r_*\,\mathfrak{Im}\,\omega} - e^{-r_*|\,\mathfrak{Im}\,\omega|}\left[1+\tfrac{k}{\epsilon}\right] + \\ &+ e^{-r_*|\,\mathfrak{Im}\,\omega|}\left[1+\tfrac{k}{\epsilon}\right]\exp\left[\frac{1}{|\omega|}\int_{r_*}^{\infty} dx|V(x)|e^{x(|\,\mathfrak{Im}\,\omega|-\mathfrak{Im}\,\omega)}\right]\,.\end{aligned} \tag{4.22}$$

Thus if (4.15) holds, the Born series is absolutely convergent. Along with an analogous result for the derivative of ψ'_ω with respect to ω, the convergence again implies analyticity of ψ'_ω in the strip (4.3). This completes the proof of our claim including α' corrections.

5. Discussion

We have seen that frequency space glueball propagators in large-N thermal $\mathcal{N}=4$ SYM theory on $\mathbb{R}^{1,3}$ have different analytic structures at weak and strong 't Hooft coupling. In position space this difference translated into qualitatively different time decays of field fluctuations about equilibrium. In the perturbative regime the correlators are known to exhibit power law falloff on intermediate time scales followed by a slow exponential decay at late times. We contrasted this with the rapid pure exponential decays seen at strong coupling.

Based on our free field computations and the generally understood aspects of weakly interacting plasmas, we expect that the perturbative real time response functions will exhibit branch cut non-analyticities. These differ qualitatively from the strong coupling quasinormal poles, which we further argued survive away from the strict infinite coupling limit. This picture suggests that these propagators must depend non-analytically on the 't Hooft coupling. The non-analyticity could either be due to nonperturbative corrections to both weak and strong coupling limits or else to a phase transition at an intermediate coupling. As we noted, however, further work is needed to rule out the possibility that the branch cuts seen in perturbation theory actually split into a series of poles in the large-N theory, thus allowing a continuous interpolation to strong coupling.

One observation that may favour the phase transition possiblity is that to each order in the large-N expansion, the perturbative 't Hooft expansion for zero temperature $\mathcal{N}=4$ SYM theory is thought to be regular at $\lambda = 0$, with a finite radius of convergence. This is due to the slow growth of the number of planar diagrams at each order in λ [32]. Including a finite temperature introduces a dependence on $\sqrt{\lambda}$ but may also spoil the regularity because of infrared divergences in perturbation theory. These divergences can be regularised by introducing an IR cutoff at the nonperturbative magnetic scale (λT) and result in a $\log(\lambda^{-1})$ dependence in perturbation theory. It is conceivable that the nonperturbative physics of magnetic screening could also have additional effects at weak coupling such as breaking the branch cuts into poles, particularly in the large-N limit. This would be a very interesting novel phenomenon in Yang-Mills plasma physics that deserves to be further investigated. If the non-analyticity is not due to nonperturbative corrections, then our results predict a phase transition in finite temperature Yang-Mills theory on $\mathbb{R}^{1,3}$.

Regarding corrections to the strong coupling results, we have argued that perturbative α' corrections do not lead to a broadening or merging of the poles into branch cuts. In principle there could be nonperturbative α' corrections to the strong coupling results, but there are no obvious worldsheet instantons in the bulk. Our strong coupling considerations have all been in the strict large-N limit. It is possible that the quasinormal poles are broadened into branch cuts by $1/N$ effects and it is expected that $1/N$ corrections will introduce additional branch cuts in real time correlators at all couplings [25].

A natural candidate for a dual phase transition in the bulk is given by the Horowitz-Polchinski correspondence [33, 34]. As the AdS background becomes increasingly curved, the horizon size eventually becomes of the order of the string scale. The Horowitz-Polchinski correspondence suggests that in this regime the appropriate description of the system is as an ensemble of highly excited string states. It would be interesting to ascertain whether there is a connection between the qualitative change in behaviour we find in field theory from strong to weak coupling and the putative change in the bulk from a black hole to string states.

These possibilities have been entertained before. A phase transition at a critical 't Hooft coupling in finite temperature $\mathcal{N} = 4$ SYM theory was previously argued for in [35], mainly on the basis of analytic properties of the free energy as a function of coupling. It was then suggested that this phase transition was dual to the Horowitz-Polchinski point in [36].[2] A key question for placing the arguments about phase transitions on a firmer footing is the identification of an appropriate order parameter. One would also want to establish whether the phase transition existed only in dynamical response functions or whether the static equilibrium theory was also affected.

Finally, in a different direction, various proposals have been made for isolating features of the bulk black hole singularity in dual field theory correlators [37, 38]. These works have focused on infinitely massive bulk probes in order to employ a semiclassical approximation. The correlators we have computed in field theory are dual to massless minimally coupled bulk scalars. Nonetheless, it would be interesting to see if signatures of a singularity, possibly resolved, can be seen in our expressions.

[2] This paper further argued that the Hawking-Page transition would not continue to weak coupling; but this now appears not to be the case [2, 3]. However, that argument seems to be independent of the arguments in the same paper and in [35] for a phase transition as a function of the 't Hooft coupling.

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A. Effect of thermal mass and width

We discuss how the inclusion of the thermal mass $m_T \sim \sqrt{\lambda}T$ and thermal width Γ for the intermediate gluon states in the correlator (2.6) changes the analytic structure of the frequency space propagator. Including a mass changes the frequencies in the real time propagators (2.4) to $E = \omega_p \equiv \sqrt{p^2 + m_T^2}$. Restricting attention to the homogeneous case (the inhomogeneous case is similar) the retarded glueball propagator becomes

$$\langle \mathcal{G}^0(t)\mathcal{G}^0(0)\rangle_R = -4\frac{N^2}{\pi^2}\Theta(t)\int_0^\infty p'^2 dp' \frac{(2p'^2 + m_T^2)^2}{\omega_{p'}^2} \coth\left(\frac{\omega_{p'}}{2T}\right)\sin(2\omega_{p'}t)\,, \tag{A.1}$$

where $\omega_{p'} = \sqrt{p'^2 + m_T^2}$. Upon Fourier transforming to frequency space this function is seen to have branch cuts. The easiest way to see this is to note that upon Fourier transforming $\Theta(t)\sin(2\omega_{p'}t)$ using an $i\epsilon$ prescription we get

$$G_R^0(\omega) = 2\frac{N^2}{\pi^2}\int_0^\infty p'^2 dp' \frac{(2p'^2 + m_T^2)^2}{\omega_{p'}^2} \coth\left(\frac{\omega_{p'}}{2T}\right)\left[\frac{1}{\omega - 2\omega_{p'} + i\epsilon} - \frac{1}{\omega + 2\omega_{p'} + i\epsilon}\right]. \tag{A.2}$$

The presence of discontinuities can now be inferred from the imaginary parts of the energy denominators in the above expression by using the representation

$$\frac{1}{\omega \pm 2\omega_{p'} + i\epsilon} = P\frac{1}{\omega \pm 2\omega_{p'}} + i\pi\delta(\omega \pm 2\omega_{p'})\,. \tag{A.3}$$

It is now a straightforward excercise to see that (A.2) has a square root branch cut discontinuity between $\omega = 2m_T$ and $\omega = -2m_T$. Note that this cut on the real axis would also appear for a massive particle at zero temperature.

Finally, we turn to the inclusion of the thermal width in the propagators. This is a subtle issue since it involves unstable intermediate states. We can provide a heuristic argument for its effect by simply looking at the form of the propagators for a massive but unstable resonance at zero temperature. The propagator for such a resonance, at zero temperature, is $(p_0^2 - p^2 - m^2 + i\Gamma)^{-1}$ where Γ is the width of the resonance. Repeating the above excercise in the real time domain and computing the frequency space retarded propagator, it is clear that the branch cut along the real axis gets shifted off the real axis. Whether this conclusion is correct depends on a more detailed and precise field theoretical analysis of the analytic structure of self-energy corrections to the internal lines.

B. Thermal glueball propagators on $\mathbb{R} \times S^3$

In this appendix we indicate how our field theory computations may be generalised to thermal Yang-Mills theory on $\mathbb{R} \times S^3$. The glueball time dependence in this context is interesting as it may be a formalism in which one can study nonequilibrium processes such as black hole evaporation.

B.1 Expansion in harmonics

Free Yang-Mills theory on $\mathbb{R} \times S^3$ has a nontrivial phase structure because the finite spatial volume restricts the partition function to a sum over colour singlets [2, 3]. As emphasised in [3] this may be seen in the fact that the nondynamical mode

$$\alpha^a \equiv \frac{g}{\mathrm{Vol}S^3} \int_{S^3} A_0^a \, d\Omega \,, \tag{B.1}$$

is strongly coupled even in the $g^2 N \to 0$ limit and so needs to be treated nonperturbatively. The different phases are characterised by the distribution of eigenvalues of α [3].

Working on $S^3 \times \mathbb{R}$, it is useful to decompose the gauge potential into transverse vector spherical harmonics on S^3

$$A_i^a(t, \theta) = \sum_m A^{am}(t) V_i^m(\theta) \,, \tag{B.2}$$

In this notation, m denotes all the relevant quantum numbers for the harmonic decomposition. For scalar harmonics these are $\{j_m, m_m, n_m\}$ and there is an extra $\varepsilon^m = \pm 1$ for the vector harmonics [4, 39]. The quadratic part of the Yang-Mills action then becomes

$$\mathcal{L}_2 = -\frac{1}{2} A^{a\bar{m}} \left[D_t^2 + E_m^2 \right] A^{am} \,. \tag{B.3}$$

As previously, in free field we may work directly with the physical transverse degrees of freedom and ignore the other modes and ghosts. In expression (B.3)

$$D_t A^{am} = \frac{\partial A^{am}}{\partial t} + f^{abc} \alpha^b A^{cm} \,, \tag{B.4}$$

and the energy levels are

$$E_m = j_m + 1 \,. \tag{B.5}$$

We are working in units in which the radius of the S^3 is one.

The glueball operator (2.2) may also be expanded into harmonics. We restrict ourselves here to the $\mathrm{Tr}\, F^2$ operator. The quadratic term gives

$$\mathcal{G}_2 = 2 \left[-D_t A^{am} D_t A^{an} D^{mnp} + \varepsilon^m \varepsilon^n E_m E_n A^{am} A^{an} D^{mnp} \right] S^p \,. \tag{B.6}$$

This expression should also contain terms dependent on the time component A_0 of the gauge field, but these do not contribute to the free theory propagators. The expression (B.6) is obtained using completeness of the harmonic decomposition and the triple overlap [39]

$$D^{mnp} = \int_{S^3} d\Omega \, V^{im} V_i^n S^p \,, \tag{B.7}$$

where S and V are the scalar and vector harmonics, respectively.

The thermal propagator following from the action (B.3) is now, for $t > 0$,

$$\langle A^{am}(0)A^{bn}(t)\rangle = \delta^{mn}\frac{ie^{-f\alpha t}}{2E_m}\left[(\bar{n}(E_m)+1)e^{-iE_m t} + n(E_m)e^{iE_m t}\right], \tag{B.8}$$

where we have suppressed the colour indices and used the abbreviation

$$(f\alpha)^{ac} \equiv f^{abc}\alpha^b\,. \tag{B.9}$$

The Bose-Einstein distribution is now

$$n(E) = \frac{1}{e^{\beta(E-if\alpha)}-1}\,. \tag{B.10}$$

We can see in (B.10) that the α condensate plays the role of a chemical potential. This corresponds with the fact that α is a Lagrange multiplier for the conservation of colour charge.

B.2 The free theory two point function

We find that for $t > 0$ the two point function of the glueball (B.6) is given by

$$\langle \mathcal{G}^p(0)\mathcal{G}^q(t)\rangle = -\frac{\delta^{p,\bar{q}}}{2\pi^2 E_p}\,\mathrm{Tr}\left[\sum_{I_+} c^+_{mnp}\,Q^+_{mn}(t) + \sum_{I_-} c^-_{mnp}\,Q^-_{mn}(t)\right], \tag{B.11}$$

where the first sum runs over

$$I_+ = \{m,n \mid |j_m - j_n| \le j_p \le j_m + j_n - 2\}\,, \tag{B.12}$$

and the second over

$$I_- = \{m,n \mid |j_m - j_n| + 2 \le j_p \le j_m + j_n\}\,. \tag{B.13}$$

The coefficients are

$$\begin{aligned} c^+_{mnp} &= \left[(E_m+E_n+E_p)^2-1\right]\left[(E_m+E_n-E_p)^2-1\right], \\ c^-_{mnp} &= \left[(E_m+E_p-E_n)^2-1\right]\left[(E_n+E_p-E_m)^2-1\right], \end{aligned} \tag{B.14}$$

and the time dependence is contained in

$$\begin{aligned} Q^+_{mn}(t) &= (\bar{n}(E_m)+1)(n(E_n)+1)e^{-i(E_m+E_n)t} + n(E_m)\bar{n}(E_n)e^{i(E_m+E_n)t}\,, \\ Q^-_{mn}(t) &= (\bar{n}(E_m)+1)\bar{n}(E_n)e^{-i(E_m-E_n)t} + n(E_m)(n(E_n)+1)e^{i(E_m-E_n)t}\,. \end{aligned} \tag{B.15}$$

In deriving these results one needs to use identities for 3j-symbols that may be found for example in the appendices of [4].

The trace in (B.11) is over the colour indices of $f\alpha$ as given in (B.9). This may be expressed in terms of the eigenvalues λ_i of α considered in [3] as follows: For some function F of $f\alpha$ we have

$$\mathrm{Tr}\,F(f\alpha) = \sum_{i,j=1}^{N} F(\lambda_i - \lambda_j)\,. \tag{B.16}$$

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In practice, we need to evaluate this sum in the two possible vacua identified by [3]. The first is the maximally clumped vacuum in which all of the N eigenvalues are equal

$$\lambda_i = \text{const.} \quad \Rightarrow \quad \operatorname{Tr} F(f\alpha) = N^2 F(0)\,. \tag{B.17}$$

This is the correct vacuum at high temperatures. The other possibility is a uniformly distributed vacuum

$$\lambda_i = \frac{2\pi i}{N} \quad \Rightarrow \quad \operatorname{Tr} F(f\alpha) = 2N \sum_{n=1}^{N-1} F\left(\frac{2\pi n}{N}\right)\left(1-\frac{n}{N}\right) + NF(0)\,, \tag{B.18}$$

which is the correct vacuum at low temperatures. In this last expression we assumed that F is an even function which is indeed true for (B.11).

The two point function (B.11) has no mixing between the different S^3 harmonics, as we expect for a free theory. The simplest case to consider is the propagation of the homogeneous mode $p = q = 0$. In this case from (B.13) we have $E_m = E_n$ in (B.11) and only the first sum contributes to the time dependence. For the $\lambda_i = \text{const}$ vacuum the sums in (B.11) for the homogeneous mode admit a compact expression in terms of theta functions

$$\begin{aligned} \Re\langle \mathcal{G}^0(0)\mathcal{G}^0(t)\rangle &= \frac{-N^2}{2\pi^2}\left[-4\frac{d}{dt} + \frac{d^3}{dt^3}\right]\frac{d}{d\beta}\frac{\vartheta_1'(t,e^{-\beta/2})}{\vartheta_1(t,e^{-\beta/2})}\,, \\ \mathfrak{Im}\langle \mathcal{G}^0(0)\mathcal{G}^0(t)\rangle &= \frac{-N^2}{2\pi^2}\left[4\frac{d^2}{dt^2} - \frac{d^4}{dt^4}\right]\left[\frac{\vartheta_1'(t,e^{-\beta/2})}{\vartheta_1(t,e^{-\beta/2})} - \frac{\vartheta_1'(t,e^{\beta/2})}{\vartheta_1(t,e^{\beta/2})}\right]. \end{aligned} \tag{B.19}$$

These expressions are true for $t > 0$. We have dropped various $\delta(t)$ contributions to the real part. We have not attempted to generalise (B.19) to the inhomogeneous modes.

One significant feature of the result given in expressions (B.11) to (B.15) is that it is periodic in time. The periodicity of (B.11) under $t \to t + 2\pi$ is a special case of the quasi-periodicity that one always finds for field theories on a compact spatial manifold. The exact periodicity in this case arises because in our free theory on $S^3 \times \mathbb{R}$ the energy eigenvalues are evenly spaced. The quasi-periodicity is expected to be dual to large gravitational fluctuations in the bulk [40] such as the formation and evporation of black holes. However, this correspondence has only been made precise when averaged over long times [41, 42]. The requirement of long time averaging is due to the thermal nature of the semiclassical approximation in gravity.

At increasingly high temperatures, $\beta \to 0$, the nontrivial time dependence is restricted to times $t \lesssim \beta$, much shorter than the full period. This limit, which can be viewed as letting $S^3 \to \mathbb{R}^3$, is simpler to study analytically was the subject of the prvious three sections. Note however that the nontrivial phase structure uncovered in [2, 3] occurs at finite temperatures. In particular, addressing black hole evaporation in this formalism will require working at finite temperature.

At low temperatures, $\beta \to \infty$, we need to use the vacuum with evenly spaced eigenvalues [3]. Thus we use the formula (B.18) to evaluate the traces. In particular, evaluating

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the leading finite temperature correction will require the result

$$\mathrm{Tr}\cos\beta f\alpha = \frac{\sin^2\beta\pi}{\sin^2\frac{\beta\pi}{N}}\,. \tag{B.20}$$

The resulting two-point correlator is then

$$\langle\mathcal{G}(0)\mathcal{G}(t)\rangle = -\frac{i2N^2}{\pi^2\sin^5 t}\left[\cos t + 2\cos^3 t\right] - \frac{\sin^2\beta\pi}{\pi^2\sin^2\frac{\beta\pi}{N}}e^{-2\beta}e^{-i4t} + \cdots\,. \tag{B.21}$$

The retarded propagator is then

$$\langle\mathcal{G}(0)\mathcal{G}(t)\rangle_R = -\frac{4N^2}{\pi^2\sin^5 t}\left[\cos t + 2\cos^3 t\right] + 2e^{-2\beta}\frac{\sin^2\beta\pi}{\pi^2\sin^2\frac{\beta\pi}{N}}\sin 4t + \cdots\,. \tag{B.22}$$

The leading finite temperature correction is exponentially suppressed.

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